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Some Fixed Point Theorem in Dislocated Metric Space

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ABSTRACT: In this paper we established a common fixed point theorem for two pairs of weakly compatible maps in dislocated metric space, which generalizes and improves similar fixed point results.

I. INTRODUCTION

Fixed point theory is one of the most dynamic research subjects in non-linear science. In 1922, S. Banach proved a fixed point theorem for contraction mapping in metric space. In 2000, Hitzler [9] introduced the concept of dislocated metric space, a generalization of metric space and presented variants of Banach contraction principle in such space.

Bennani et al [ ] established two new common fixed point theorem for four self-maps on dislocated metric space, which improve the results of Panthi and Jha [7].Dislocated metric space plays very important role in topology, logical programming and in electronic engineering. C.T Age and J.N. Salunke [2], A. Isufati [1] established some important fixed point theorems in single and pair of mapping in single and pair of mapping in dislocated spaces.

II. PRELIMINARIES

Now we start with the following definitions, lemmas and theorems.

Definition 2.1 Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(i) $d(x, y) = d(y, x)$

(ii) $d(x, y)= d(y, x) = 0$ implies $x = y$.

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called dislocated metric(or simply d-metric) on X.

Definition 2.2 A sequence $\{x_n\}$ in a d-metric space (X, d) is called a Cauchy sequence if for given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \varepsilon$.

Definition 2.3 A sequence in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, x is called limit of $\{x_n\}$ (in d)and we write $x_n \to x$

Definition 2.4 [8] A d-metric space (X, d) is called complete if every Cauchy sequence in it is convergent with respect to d.

Definition 2.5 Let (X, d) be a d-metric space. A map $T : X \to X$ is called contraction if there exists a number λ with $0 \leq \lambda < 1$ such that $d(T x, T y) \leq \lambda\, d(x, y)$.

We state the following lemmas without proof.

Lemma 2.6-Let (X, d) be a d-metric space. If $T : X \to X$ is a contraction function, then $\{T_n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.

Lemma 2.7 -Limits in a d-metric space are unique.

Definition 2.8 Let A and S be mappings from a metric space (X, d) into itself. Then, A and S are said to be weakly compatible if they commute at their coincident point; that is, $Ax = Sx$ for some $x \in X$ implies $ASx = SAx$.

Theorem 2.9 Let (X, d) be a complete d-metric space and let $T : X \to X$ be a contraction mapping, then T has a unique fixed point.

III. MAIN RESULTS

Theorem 3.1- Let (X,d) be a complete d −metric space. Let E,F,G and H are four self maps of X, satisfying the following condition –

(i) $H(X) \subseteq E(X) and\ G(X) \subseteq F(X)$

(ii) The pair $(G,E) and\ (H,F)$ are weakly compatible and

(iii) $d(G_x,H_y) \le \alpha\{d(E_x,H_y)+d(F_y,G_x)\}+\beta\{d(F_y,H_y)+d(F_y,G_x)\}+ \gamma\{d(E_x,F_y)\}+\delta\{d(G_x,F_y)$

For all $x,y \in X$ $where$ $\alpha,\beta,\gamma,\delta \ge 0,\ 0 \le 3\alpha+2\beta+\gamma+2\delta < \frac{1}{2}$

Then E,F,G,H have a unique fixed point.

Proof. Using the condition (i), we define sequence $\{x_n\}$ and $\{y_n\}$ in X, then

$y_n = Gx_n = Fx_{n+1}$ and$y_{n+1} = Hx_{n+1} = Ex_{n+2}$,n=0,1,2,3,4,……….

If $y_n = y_{n+1}$ for some n, then $Fx_{n+1} = Hx_{n+1}$.Therefore x_{n+1} is a coincidence point of F and H. Also$y_{n+1} = y_{n+2}$ for some n,then$Gx_{n+2} = Ex_{n+2}$.Hence x_{n+2} is a coincidence point of G and Assume that $y_n \ne y_{n+1}$ for all n.

Then By (iii),

$$d(y_n,y_{n+1}) = d(Gx_n,Hx_{n+1})$$
$$\le \alpha\{d(Ex_n,Hx_{n+1})+ d(Fx_{n+1},Gx_n)\}+\beta\{d(Fx_{n+1},Hx_{n+1})+ d(Fx_{n+1},Gx_n)\}+\gamma\{d(Ex_n,Fx_{n+1})\} + \delta\{d(Gx_n,Fx_{n+1})\}$$
$$\le \alpha\{d(y_{n-1},y_{n+1})+d(y_n,y_n)\}+\beta\{d(y_n,y_{n+1})+d(y_n,y_n)\}+ \gamma\{d(y_{n-1},y_n)\}+\delta\{d(y_n,y_n)\}$$
$$\le \alpha\{[d(y_{n-1},y_n)+d(y_n,y_{n+1})]+[d(y_n,y_{n-1})+d(y_{n-1},y_n)\} + \beta\{d(y_n,y_{n+1})+[d(y_n,y_{n-1})+d(y_n,y_{n-1})\}+ \gamma\{d(y_{n-1},y_n)\} + \delta\{d(y_n,y_n)\}$$
$$d(y_n,y_{n+1})[1-\alpha-\beta] \le d(y_{n-1},y_n)[3\alpha+2\beta+\gamma+2\delta]$$
$$d(y_n,y_{n+1}) \le \frac{3\alpha+2\beta+\gamma+2\delta}{[1-\alpha-\beta]}\ d(y_{n-1},y_n)$$
$$d(y_n,y_{n+1}) \le S\ d(y_{n-1},y_n)$$

Where $S = \frac{3\alpha+2\beta+\gamma+\delta}{[1-\alpha-\beta]} < 1$

This shows that,

$d(y_n,y_{n+1}) \le S\ d(y_{n-1},y_n) \le S^2 d(y_{n-2},y_{n-1}) \dots\dots\dots\dots\dots. \le S^n\ d(y_0,y_1)$

For every integer $p > 0, We\ have$

$$d(y_n,y_{n+p}) \le d(y_n,y_{n+1})+ d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+\cdots\dots\dots+ d(y_{n+p-1},y_{n+p})$$
$$\le (1+S+S^2+\cdots\dots\dots\dots\dots.+S^{p-1})d(y_n,y_{n+1})$$
$$\le \frac{S^n}{1-S} d(y_0,y_1)$$

Since $0 < S < 1, S^n \to 0\ as\ n \to \infty$

So we get $d(y_n,y_{n+p}) \to 0. this\ implies\ \{y_n\}$ is a Cauchy sequence in a complete dislocated metric space. So there exist a point $z \epsilon X$ such that$\{y_n\} \to z$.

Therefore, the sub sequences,

$\{Gx_n\} \to z, \{Fx_{n+1}\} \to z, \{Hx_{n+1}\} \to z, \{Ex_{n+2}\} \to z$

Since$H(X) \subseteq E(X)$, then there exist $u \in X$ such that $u \in X\ such\ that\ z = Eu$.

So,

$$d(Gu,z) = d(Gu,Hx_{n+1})$$
$$\le \alpha\{d(Eu,Hx_{n+1})+d(Gu,Fx_{n+1})\}+\beta\{d(Fx_{n+1},Hx_{n+1})+ d(Fx_{n+1},Gu)\}+ \gamma\{d(Eu,Fx_{n+1})\} + \delta\{d(Gu,Fx_{n+1})\}$$

As$n \to \infty$,

$$\le \alpha\{d(z,z)+d(Gu,z)\}+\beta\{d(z,z)+ d(z,Gu)\}+\gamma\{d(z,z)\}+\delta\{d(Gu,z)\}$$

$d(Gu,z) \le 2[\alpha+\beta+\gamma]d(Gu,z)+[\alpha+\beta+\delta]d(Gu,z)$

$d(Gu,z) \le [3\alpha+3\beta+2\gamma+\delta]\ d(Gu,z)$

This is a contradiction, since$[3\alpha+3\beta+2\gamma+\delta] < 1$

So we have $Gu = Eu = z$.
Again, since $G(X) \subseteq F(X)$, there exist a point $v \in X$, such that $z = Fv$

$$d(z,Hv) = d(Gu,Hv)$$
$$\leq \alpha\{d(Eu,Hv) + d(Gu,Fv)\} + \beta\{d(Fv,Hv) + d(Fv,Gu)\} + \gamma\{d(Eu,Fv)\} + \delta\{d(Fv,Gu)\}$$
$$\leq \alpha\{d(z,Hv) + d(z,z)\} + \beta\{d(z,Hv) + d(z,z)\} + \gamma\{d(z,z)\} + \delta\{d(z,z)\}$$
$$d(z,Hv) \leq 2[\alpha+\beta+\gamma+\delta]d(z,Hz) + d(z,Hv)[\alpha+\beta]$$
$$d(z,Hv) \leq [3\alpha+3\beta+2\gamma+2\delta]\, d(z,Hv)$$

This is a contradiction, since $[3\alpha+3\beta+\gamma+\delta] < 1$.
so We have $Gu = Hv = Eu = Fv = z$
Since the pair (G,E) *are* weakly compatible so by definition $GEu=EGu$ implies Gz = Ez.
Now next we show that z is the fixed point of G.if $Gz \neq z$, *then*

$$d(Gz,z) = d(Gz,Hv)$$
$$\leq \alpha\{d(Gz,z) + d(Gz,Fv)\} + \beta\{d(z,z) + d(z,Gz)\} + \gamma\{d(Gz,z)\} + \delta\{d(z,Gz)\}$$
$$\leq \alpha\{d(Gz,z) + d(Gz,z)\} + \beta\{d(z,z) + d(z,Gz)\} + \gamma\{d(Gz,z)\} + \delta\{d(z,Gz)$$
$$d(Gz,z) \leq [2\alpha+3\beta+\gamma+\delta]d(z,Gz)$$

Which is a contradiction, So we have $Gz = z$
This implies that $Gz = Ez = z$.
Similarly we can show that Hz = $Fz = z$.
Hence $Gz = Ez = Hz = Fz = z$.
This show that z is the common fixed point of the mapping E, F, G, H.
Uniqueness *– Let* $u \neq v$ *be two common fixed point of the mapping E,F,G,H. then*
We have,

$$d(u,v) = d(Gu,Hv)$$
$$\leq \alpha\{d(Eu,Hv) + d(Gu,Fv)\} + \beta\{d(Fv,Hv) + d(Fv,Gu)\} + \gamma\{d(Eu,Fv)\} + \delta\{d(Fv,Gu)\}$$
$$\leq \alpha\{d(u,v) + d(u,v)\} + \beta\{d(v,v) + d(v,u)\} + \gamma\{d(u,v)\} + \delta\{d(v,u)\}$$
$$\leq [2\alpha+3\beta+\gamma+\delta]d(u,v)$$

Which is contradiction.
This shows that $d(u,v) = 0$, So we have $u = v$.
Corollary *– Let (X,d) be a complete d-metric space .Let* $G,H: X \to X$ *be continuous mapping* satisfying ,

$$d(G_x,H_y) \leq \alpha\{d(E_x,H_y) + d(F_y,G_x)\} + \beta\{d(F_y,H_y) + d(F_y,G_x)\} + \gamma\{d(E_x,F_y)\} + \delta\{d(G_x,F_y)$$

For all $x,y \in X, where\ \alpha,\beta,\gamma,\delta \geq 0, 0 \leq \alpha+3\beta+\gamma+\delta < 1$. Then G,H have a unique common fixed point.
Proof-If we take E = F= I an identity mapping in the above theorem and follow the similar proof as that in theorem. We can establish this corollary.

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